DYNAMICS OF THE PIANOFORTE STRING AND THE HAMMER

PART IV (STUDY OF DURATION OF IMPACT)

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ABSTRACT. The complete dynamics of the finite Pianoforte string and the hammer of any kind has been worked out in Parts I, II and III. In this part only expressions for the the duration of contact when the hammer strikes at different points of a finite string have been worked out. The comparison of the theoretical values, with those obtained—experimentally by different authors, is made graphically, and remarkable agreement is found in each case. It is found from the theory, that the duration of contact changes discontinuously with distance and for the higher values of the 'mass-ratio' the curve shifts towards the greater values of the 'duration' as was found experimentally by George, Kar-Ghosh and others. The comparison of the theoretical values with those obtained experimentally by M. Ghosh and Banerjee-Ganguli is made graphically and it is found that the agreement is remarkable, even when a massive hammer is used, *i.e.*, for a large 'mass-ratio' where Das and other theories fail completely. The variation of the duration of contact with different values of the elastic constant of the hammer, for a given striking distance, as is obtained from the theory, is also compared graphically with that obtained experimentally. There is very good agreement in this case.

INTRODUCTION

The dynamics of the Pianoforte string and the hammer has been developed completely[1] in Parts I, II and III. The theory has been tested in different ways. The pressure-time curve, for hard and elastic hammers, has been drawn for different cases. It is found that, in case of the hard hammer, the pressure changes periodically in a discontinuous manner. As the hard hammer is replaced by an elastic hammer, the discontinuous periodic rises of pressure lose their sharp angularities and humps become less and less pronounced. Similar behaviour of pressure-time curve for different values of the elastic constant was observed recently by Davy[2] and his co-workers. The fact that the shorter segment of the string sets in vibration as soon as the hammer comes in contact with the string was observed previously. No fruitful explanation of the phenomenon was, however, known. The present theory explains the same fully, as is evident from the time-displacement curve given in one of the previous papers.[3] The duration of impact of the hammer has also been previously calculated for a few cases. In this paper an elaborate and systematic study of the duration of contact for hard as well as elastic hammer will be made, and the calculated values will be compared with those obtained experimentally.

The symbols to be used in the present paper have got the same meaning as in Parts I, II and III.

The expression for the pressure exerted by the hammer as derived previously is a function of time. And, as already pointed out, the duration of contact, Φ, *is the lowest positive root* obtained by solving $P_n(t)=0$.

HARD HAMMER

(i) *When the hammer strikes very near the end*, so that $\frac{a}{c}$ is very small, the pressure exerted by the hammer is given by [*vide* eq. (9), Part 1]

$$P=mv_0\frac{\mu^2+\nu^2}{\nu}e^{-\mu t}\sin\left(\nu t-\tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2}\right).$$

and the duration of impact is given by

$$\Phi=\frac{1}{\nu}\tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2},$$

where

$$\mu=\frac{M_1}{m_0\theta_1},\qquad \nu=\frac{1}{\theta}\sqrt{\frac{4M_1}{m_0}-\frac{M_1^2}{m_0^2}}.$$

The general expression of the pressure exerted by the hammer during contact is [*vide* eq. (56), Part III]

$$-P=\frac{2T}{c}\Big[f'_1(t)+\sum_1^n f'_{n+1}(t-n\theta_1)+\sum_1^n f'_{n+1}(t-n\theta_2)+2\{f_3'-f_2'\}(t-\Theta)$$
$$+\sum_1^n\{(n+2)f'_{n+3}-2(n+1)f'_{n+2}+nf'_{n+1}\}\{(t-\Theta-n\theta_1)+(t-\Theta-n\theta_2)\}$$
$$+2\{3f_5'-6f_4'+4f_3'-f_2'\}(t-2\Theta)\Big] \qquad \ldots (56)$$

where $f'_n(t)=$

$$A^n v_0\Bigg[e^{-qt}.\sum_{r=1}^n(-)^{n+r-2}.\frac{\Gamma(n+r-1)}{\Gamma(n)\Gamma(r)}\beta^{r-1}\Big\{q^{n-1}.\frac{t^{n-r}}{(n-r)!}-{}^{n-1}C_1.q^{n-2}.\frac{t^{n-r-2}}{(n-r-2)!}$$
$$+{}^{n-1}C_2.q^{n-3}\frac{t^{n-r-3}}{(n-r-3)!}-+\Big\}+(-)^{2n-1}e^{-pt}\sum_{r=1}^n\frac{\Gamma(n+r-1)}{\Gamma(n)\Gamma(r)}.\beta^{r-1}.$$
$$\Big\{p^{n-1}\frac{t^{n-r}}{(n-r)!}-{}^{n-1}C_1.p^{n-2}\frac{t^{n-r-2}}{(n-r-2)!}+{}^{n-1}C_2.p^{n-3}.\frac{t^{n-r-3}}{(n-r-3)!}-+\Big\}\Bigg] \qquad \ldots (56a)$$

when $q \neq p$ and $\beta = \frac{1}{p-q}$, $A = \beta(q+p)$; so

and $$2\beta p = (A+1),\ 2\beta q = (A-1)$$

and $$f'_n(t) = 2^n v_0 e^{-qt} \left\{ \frac{(qt)^n}{n!} - {}^{n-1}C_1 \frac{(qt)^{n+1}}{(n+1)!} + - + (-)^{n-1} \frac{(qt)^{2n-1}}{(2n-1)!} \right\} \quad \dots \quad (56b)$$

when $q = p$.

For hard hammer we put $E \to \infty$ in eq. (56a). From the above pressure equation, we are able to calculate the value of the duration of contact, for the particular case we require.

(ii) *When the striking distance a is finite and the other segment of the string b is large enough* compared to a so that no wave after suffering reflection from the end $x=l$, arrives at the hammer before the pressure terminates. The expressions for the pressure at different intervals are [*vide* eq. (13), Part I or eqs. (56) and (56a) for $b \backsim l \backsim \infty$ and $E \to \infty$]

$$P_1 = \frac{2T}{c} f'_1(t) = 2\rho v_0 c e^{-qt}$$

$$P_2 = \frac{2T}{c} \left\{ f'_1(t) - f_2'(t_1) \right\} = P_1 + 2\rho v_0 c e^{-qt} (1 - qt_1)$$

$$\dots \qquad \dots \qquad \dots \qquad \dots$$

$$P_{n+1} = P_n + \frac{2T}{c} f'_{n+1}(t_n)$$

$$= P_n + 2\rho v_0 c e^{-qt_n} \left\{ 1 - {}^nC_1 . qt_n + {}^nC_2 \frac{q^2 t_n^2}{2!} - + (-)^n \, {}^nC_n \frac{q^n t_n^n}{n!} \right\}. \quad \dots \quad (69)$$

We find, from above, that $P_1 = 0$ has no finite root except at $t = \infty$ so that impact must not terminate during the first interval, $\theta_1 > t > 0$.

If the pressure terminates during the second interval, we get by solving $P_2 = 0$

$$\frac{\Phi}{\theta_1} = 1 + \frac{m}{4M_1} \left\{ 1 + e^{-\frac{4M_1}{m}} \right\} \quad \dots \quad (70)$$

provided the maximum *mass-ratio* (*i.e.*, for $\Phi = 2\theta_1$) is given by

$$1 - \frac{m}{4M_1} \left\{ 1 + e^{-\frac{4M_1}{m}} \right\} = 0 \quad \dots \quad (70.1)$$

which has a root $\frac{m}{M_1} = 3 \cdot 15$.

Thus the pressure terminates during $2\theta_1 > t > \theta_1$ provided $\frac{m}{M_1} \not> 3{\cdot}15$, *i.e.*, $a \not< 0{\cdot}32\frac{m}{\rho}$.

If the pressure terminates during $3\theta_1 > t > 2\theta_1$ we get from the equation $P_3 = 0$

$$\left(\frac{\Phi}{\theta_1} - 2\right)^2 - \gamma_{2\cdot1}\left(\frac{\Phi}{\theta_1} - 2\right) + \gamma_{2\cdot2} = 0, \qquad \dots \quad (71)$$

where

$$\gamma_{2\cdot1} = \frac{m}{2M_1}\left(2 + e^{-\frac{4M_1}{m}}\right), \qquad \dots \quad (71.1)$$

$$\gamma_{2\cdot2} = \frac{m^2}{8M_1^2}\left[1 + e^{-\frac{4M_1}{m}}\left(1 - \frac{4M_1}{m}\right) + e^{-\frac{8M_1}{m}}\right], \qquad \dots \quad (71.2)$$

provided the maximum *mass-ratio* (*i.e.*, $\Phi = 3\theta_1$) is given by

$$1 - \gamma_{2\cdot1} + \gamma_{2\cdot2} = 0. \qquad \dots \quad (71.3)$$

Duration of contact is obtained by solving eq. (71).

The equation (71.3) has root $\frac{m}{M_1} = 6{\cdot}1$ approximately, so that the impact terminates during this interval provided $\frac{m}{M_1} \not> 6{\cdot}1$, *i.e.*, $a \not< 0{\cdot}16\frac{m}{\rho}$.

If the pressure terminates during $4\theta_1 > t > 3\theta_1$ we get from the eqation $P_4 = 0$

$$\left(\frac{\Phi}{\theta_1} - 3\right)^3 - \gamma_{3\cdot1}\left(\frac{\Phi}{\theta_1} - 3\right)^2 + \gamma_{3\cdot2}\left(\frac{\Phi}{\theta_1} - 3\right) - \gamma_{3\cdot3} = 0, \qquad \dots \quad (72)$$

where

$$\gamma_{3\cdot1} = \frac{3m}{4M_1}\left(3 + e^{-\frac{4M_1}{m}}\right), \qquad \dots \quad (72.1)$$

$$\gamma_{3\cdot2} = \frac{3}{8}\frac{m^2}{M_1^2}\cdot\left[e^{-\frac{8M_1}{m}} + e^{-\frac{4M_1}{m}}\left(2 - \frac{4M_1}{m}\right) + 3\right] \qquad \dots \quad (72.2)$$

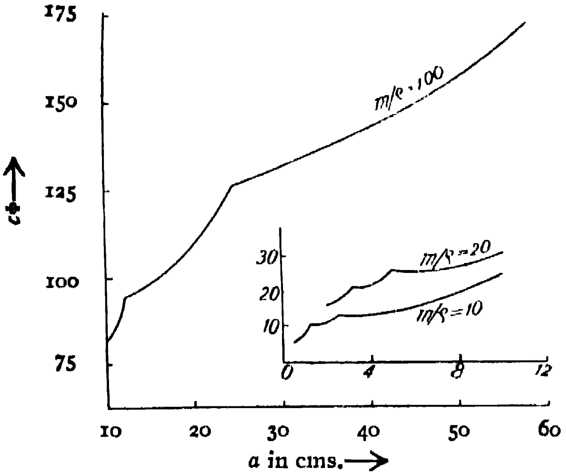
$$\gamma_{3\cdot3} = \frac{3m^3}{32M_1^3}\left[e^{-\frac{12M_1}{m}} + e^{-\frac{8M_1}{m}}\left(1 - \frac{8M_1}{m}\right) + e^{-\frac{4M_1}{m}}\left(1 - \frac{8M_1}{m} + \frac{8M_1^2}{m^2}\right) + 1\right], \qquad \dots \quad (72.3)$$

provided the maximum *mass-ratio* is given by, as is obtained by putting $\Phi = 4\theta_1$ in eq (72),

$$1 - \gamma_{3\cdot1} + \gamma_{3\cdot2} - \gamma_{3\cdot3} = 0. \qquad \dots \quad (72.4)$$

Duration of impact is obtained by solving eq. (72). The eq. (72·4) has root $m/M_1 = 10{\cdot}0$ approximately ; so the impact terminates during this interval provided $\frac{m}{M_1} \not> 10$, *i.e.*, $a \not< 0{\cdot}1\frac{m}{\rho}$.

In this way we can find out the expression for the duration of contact, when the impact terminates during any interval.



[FIGURE 1

Figure 1 shows the variation of the duration of contact with striking distance for different values of $\frac{m}{\rho}$ as predicted by the theory. We find that the duration of contact changes discontinuously with distance, and for higher values of $\frac{m}{\rho}$, the curve shifts towards the greater value of the duration of contact. The same was found experimentally by George[6], Kar-Ghosh[8], Banerjee-Ganguli[5] and others.

(*iii*) *The hammer strikes at the mid-point* $l=2a$. Equating $P_1=0$ as given by eq. (17.1), Part I, we find that the positive root other than zero is $t=\infty$, so the pressure in any case will not terminate during the first interval $\theta_1>t>0$.

If the pressure terminates during the interval $2\theta_1>t>\theta_1$, we get from $P_2=0$ as given by eq. (17.2).

$$\frac{\Phi}{\Theta}=\frac{1}{2}+\frac{m}{8M}\left(2+e^{-\frac{2M}{m}}\right). \quad \cdots \quad (73)$$

This is valid, so long as the maximum *mass-ratio* $\frac{m}{M}$ does not exceed 1·7, which is obtained as the root of the equation

$$\frac{m}{4M}\left(2+e^{-\frac{2M}{m}}\right)=1. \quad \cdots \quad (73.1)$$

When the pressure terminates during $3\theta_1 > t > 2\theta_1$ we get from $P_3 = 0$ [*vide* eq. (17.3)]

$$(\Phi-\Theta)^2 - \overset{*}{\gamma}_{2\cdot1}\frac{\Theta}{2}(\Phi-\Theta) + \overset{*}{\gamma}_{2\cdot2}\frac{\Theta^2}{4} = 0, \qquad \dots (74)$$

where
$$\overset{*}{\gamma}_{2\cdot1} = \frac{m}{2M}\left(3 + e^{-\frac{2M}{m}}\right), \qquad \dots (74.1)$$

$$\overset{*}{\gamma}_{2\cdot2} = \frac{m^2}{4M^2}\left[1 + \tfrac{1}{2}e^{-\frac{4M}{m}} + e^{-\frac{2M}{m}}\left(1 - \frac{2M}{m}\right)\right], \qquad \dots (74.2)$$

provided the maximum *mass-ratio* $\frac{m}{M}$ is given by the equation,

$$1 - \overset{*}{\gamma}_{2\cdot1} + \overset{*}{\gamma}_{2\cdot2} = 0 \qquad \dots (74.3)$$

which has a root 4.1 approximately; so the impact terminates during $3\theta_1 > t > 2\theta_1$ when $\frac{m}{M}$ lies between 1.7 and 4.1.

When the pressure terminates during $4\theta_1 > t > 3\theta_1$ we get from $P_4 = 0$ [*vide* eq. (17.4)]

$$\left(\Phi - \frac{3\Theta}{2}\right)^3 - \overset{*}{\gamma}_{3\cdot1}\frac{\Theta}{2}\left(\Phi - \frac{3\Theta}{2}\right)^2 + \overset{*}{\gamma}_{3\cdot2}\frac{\Theta^2}{4}\left(\Phi - \frac{3\Theta}{2}\right) - \overset{*}{\gamma}_{3\cdot3}\frac{\Theta^3}{8} = 0, \qquad (75)$$

where
$$\overset{*}{\gamma}_{3\cdot1} = \frac{3m}{4M}\left(4 + e^{-\frac{2M}{m}}\right), \qquad \dots (75.1)$$

$$\overset{*}{\gamma}_{3\cdot2} = \frac{3m^2}{8M^2}\left[5 + e^{-\frac{2M}{m}}\left(3 - \frac{4M}{m}\right) + 2e^{-\frac{4M}{m}}\right], \qquad \dots (75.2)$$

$$\overset{*}{\gamma}_{3\cdot3} = \frac{3m^3}{16M^3}\left[1 + e^{-\frac{2M}{m}}\left(1 - \frac{6M}{m} + \frac{4M^2}{m^2}\right) + e^{-\frac{4M}{m}}\left(1 - \frac{4M}{m}\right) + \tfrac{1}{3}e^{-\frac{6M}{m}}\right], \qquad \dots (75.3)$$

provided the maximum *mass-ratio* is given by the equation

$$1 - \overset{*}{\gamma}_{3\cdot1} + \overset{*}{\gamma}_{3\cdot2} - \overset{*}{\gamma}_{3\cdot3} = 0 \qquad \dots (75.4)$$

which has a root $\frac{m}{M} = 7{\cdot}3$ approximately. Thus the pressure terminates during this interval, so long as $\frac{m}{M} \not< 4{\cdot}10$ and $\not> 7{\cdot}30$.

During the interval $5\theta_1 > t > 4\theta_1$ by putting $P_5 = 0$ [*vide* eq. (17.5)], the duration of contact Φ is obtained as the root of the equation

$$(\Phi - 2\Theta)^4 - \overset{*}{\gamma}_{4\cdot 1}\frac{\Theta}{2}(\Phi - 2\Theta)^3 + \overset{*}{\gamma}_{4\cdot 2}\frac{\Theta^2}{4}(\Phi - 2\Theta)^2$$

$$- \overset{*}{\gamma}_{4\cdot 3}\frac{\Theta^3}{8}(\Phi - 2\Theta) + \overset{*}{\gamma}_{4\cdot 4}\frac{\Theta^4}{16} = 0 \qquad \dots (76)$$

where,

$$\overset{*}{\gamma}_{4\cdot 1} = \frac{m}{M}\left(5 + e^{-\frac{2M}{m}}\right), \qquad \dots (76.1)$$

$$\overset{*}{\gamma}_{4\cdot 2} = \frac{3m^2}{4M^2}\left[9 + e^{-\frac{4M}{m}} + 4e^{-\frac{2M}{m}}\cdot\left(1 - \frac{2M}{m}\right)\right], \qquad \dots (76.2)$$

$$\overset{*}{\gamma}_{4\cdot 3} = \frac{3m^3}{8M^3}\left[7 + e^{-\frac{6M}{m}} + e^{-\frac{4M}{m}}\cdot\left(3 - \frac{8M}{m}\right)\right.$$

$$\left. + e^{-\frac{2M}{m}}\cdot\left(5 - \frac{16M}{m} + \frac{8M^2}{m^2}\right)\right], \qquad \dots 76.3)$$

$$\overset{*}{\gamma}_{4\cdot 4} = \frac{3m^4}{8M^4}\left[1 + e^{-\frac{8M}{m}} + e^{-\frac{6M}{m}}\left(1 - \frac{6M}{m}\right)\right.$$

$$\left. + e^{-\frac{4M}{m}}\cdot\left(1 - \frac{12M}{m} + \frac{16M^2}{m^2}\right) + e^{-\frac{2M}{m}}\left(1 - \frac{10M}{m} + \frac{16M^2}{m^2} - \frac{16M^3}{m^3}\right)\right], \qquad (76.4)$$

provided the maximum mass-ratio is given by

$$1 - \overset{*}{\gamma}_{4\cdot 1} + \overset{*}{\gamma}_{4\cdot 2} = \overset{*}{\gamma}_{4\cdot 3} + \overset{*}{\gamma}_{4\cdot 4} = 0, \qquad \dots (76\cdot 5)$$

which has a root $\frac{m}{M} = 10{\cdot}4$ approximately. The pressure terminates during this interval if $\frac{m}{M}$ lies between $7{\cdot}3$ and $10{\cdot}4$.

In this way we can find out the duration of contact for any interval higher than $5\theta_1 > t > 4\theta$.

(*iv*) *The hammer strikes at a point, little away from the mid-point, of the string, i.e., b is slightly greater than a.*

If the pressure terminates during $\theta_2 > t > \theta_1$, we get the corresponding expression for the duration of contact by solving eq. $P_2 = 0$ [*vide* eq. (60), Part III]

$$\frac{\Phi}{\Theta}=\frac{a}{l}\left[1+\frac{ml}{4\,Ma}\left(1+e^{-\frac{4\,Ma}{ml}}\right)\right], \qquad \ldots \quad (77).$$

provided the maximum mass-ratio which is obtained by putting $\Phi=\frac{2b}{c}$ is given by the equation

$$\frac{m}{4M}\left(1+e^{-\frac{4\,Ma}{ml}}\right)=\left(1-\frac{2a}{l}\right). \qquad \ldots \quad (77{\cdot}1)$$

Here we find that the maximum mass-ratio depends on $\frac{l}{a}$. The equation (77.1) is solved for different values of $\frac{b}{a}$ in order to get the maximum limit of the mass-ratio for the given interval (*vide* table 1).

TABLE I

l/a	Maximum m/M
2.2	0.36
2.4	0.65
2.6	0.79
2.8	0.93
3	1.04

If the pressure terminates during $2\theta_1 > t > \theta_2$ we get–from eq. $P_2=0$ [*vide* eq. (60), Part III]

$$\frac{\Phi}{\Theta}=\frac{1}{1+e^{-\frac{4M(b-a)}{ml}}}\left[\frac{a}{l}+\frac{m}{8M}\left(2+e^{-\frac{4\,Ma}{ml}}\right)\right]$$

$$+\frac{1}{1+e^{-\frac{4M(b-a)}{ml}}}\left[\frac{b}{l}+\frac{m}{8M}\left(2+e^{-\frac{4Mb}{ml}}\right)\right], \qquad \ldots \quad (78)$$

provided the maximum *mass-ratio* which is obtained by putting $\Phi=2\theta_1$ is given by

$$\frac{m}{4M} \cdot \frac{1}{\left\{ e^{\frac{4Ma}{ml}} + e^{\frac{4M}{m}\left(1-\frac{a}{l}\right)} \right\}} \left[1 + e^{-\frac{4Ma}{ml}}\left(1+\frac{4Ma}{ml}\right) \right.$$

$$\left. + e^{-\frac{4M}{m}\left(1-\frac{a}{l}\right)}\left(1+\frac{4Mb}{ml}\right) \right] = \frac{2a}{l}. \quad \dots \quad (78.1)$$

When $a=b$, *i.e.*, at mid-point, the eq. (78) is evidently the same as eq (73).

(*v*) *Hammar strikes at* $\frac{l}{3}$ *from one end, i.e.*, $l=3a$, $b=2a$. If the pressure terminates during $2\theta_1>t>\theta_1$ we have from eq. $P_2=0$ [*vide* eq. (60), Part III]

$$\frac{\Theta}{\theta} = \frac{1}{3} + \frac{m}{4M}\left(1+e^{-\frac{4M}{3m}}\right) \quad \dots \quad (79)$$

provided the maximum *mass-ratio* $\frac{m}{M}$ is 1·04, as is given by the equation,

$$\frac{4M}{m} = \left(1+e^{-\frac{4M}{3m}}\right). \quad \dots \quad (79.1)$$

If the impact terminates during $\Theta>t>2\theta_1$ we get from $P_4=0$, for $\theta_2=2\theta_1$ [*vide* eq. (60), Part III]

$$\left(\frac{\Theta}{\theta_1}-2\right)^2 - \bar{\gamma}_{2\cdot1}\left(\frac{\Theta}{\theta_1}-2\right) + \bar{\gamma}_{2\cdot2} = 0, \quad \dots \quad (80)$$

where, $\gamma_{2\cdot1} = \frac{3m}{2M}\left\{3+e^{-\frac{4M}{3m}}\right\}$,

$$\bar{\gamma}_{2\cdot2} = \frac{9}{8}\frac{m^2}{M^2}\left\{e^{-\frac{8M}{3m}} + e^{-\frac{4M}{3m}}\left(1-\frac{4M}{3m}\right)+2\right\}, \quad \dots \quad (80.1)$$

provided the maximum *mass-ratio* $\frac{m}{M}$ is given by (as $\Theta=3\theta_1$)

$$1-\bar{\gamma}_{2\cdot1}+\bar{\gamma}_{2\cdot2}=0, \quad \dots \quad (80.2)$$

which has a root 1.76 approximately.

The comparison of the experimental values of the variation of the duration of contact with striking distance in the case of hard hammer, as given by M. Ghosh (fig. 2) and Banerjee-Ganguli (fig. 3). is made with those given by the present theory.

The duration of contact for hard hammer (*vide* Fig. 2) of mass 21·2 gm was observed for different striking distances, at a regular interval of 10 cms. from one end of the string of length 600 cms., and of linear density 0·05 gm./cm., stretched under tension 38·5 kilograms weight. With the above data as given by M. Ghosh, the duration of contact, which is the lowest positive root of the pressure equation, is calculated theoretically. These roots are obtained by solving the pressure equations numerically. They are also checked by the graphical method.

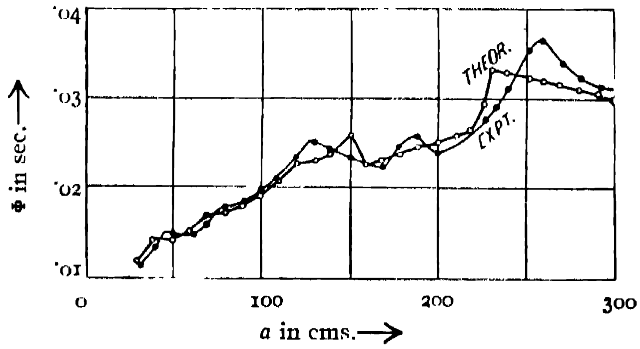


FIGURE 2

There are some striking distances at which double or triple contacts are observed. But these higher contacts, other than the first, are different from what is given by the dynamics of the struck string. These higher contacts may *slightly* modify the vibration-form of the string after impact. During impact, the hammer at first moves forward, then momentarily comes to rest, and afterwards starts moving backward, when after some time, it is separated from the string. Now, a reflected wave from either end while travelling along the string will pass through the point which was in contact with the hammer. If this reflected wave, at the instant of crossing the point of contacts, produces a displacement just equal to or greater than instantaneous separation-distance between the string and retarding hammer, then there is every possibility of fresh contact. The duration of such a contact is evidently much smaller than the first. As the velocity of wave propagation along the string is completely independent of the velocity of the hammer, so the phenomenon of multiple contact is influenced by the hammer velocity. Similar remark was also made by W. H. George. The experimental study of the same is in progress and will be published in due course.

The value of the duration of contact, when the hammer strikes the mid-point of the string, is calculated with the help of the eq. (73), as m/M is found to be less than 1·7. The values of the duration of contact, for the striking length 290 cms. down to 230 cms. are calculated with the help of eq. (78.1). Here the waves from the remoter end overtakes the hammer before it leaves the string.

Eq. (70) is used for the striking length 220 cms. down to 160 cms. as m/M is found to be less than 3·15. In these cases pressure terminates during the second interval before any wave from the remoter end overtakes the hammer. Eq. (61) is used for the striking length 150 cms. down to 90 cms. as $m/M_1 < 6{\cdot}1$. Here pressure is found to terminate during the third interval. Eq. (72) is used for the striking length 80 cms. down to 50 cms. as m/M_1 is found to be less than 10. Here the pressure is found to terminate during the fourth interval. For smaller striking lengths calculations are made graphically, where it is found that the pressure terminates during the fifth epoch.

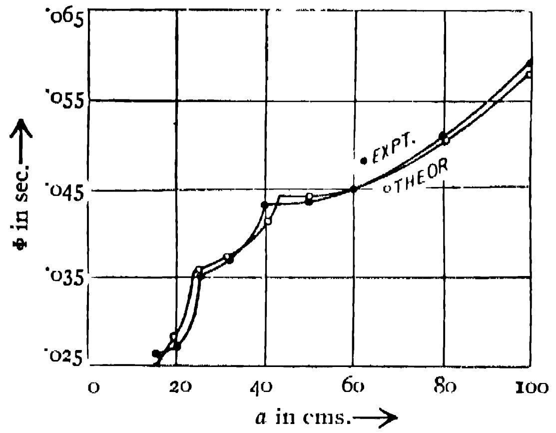


FIGURE 3

In figure 3 the values of the duration of contact when a hard hammer of mass 45 gms. striking at different points on the string of length 240 cms. and of mass 16·4 gms. stretched under a tension of 10·90 kilograms weight, are obtained by solving the pressure equations graphically. As for a massive hammer, that is, for a large mass-ratio, which is the case here, the pressure terminates after a large number of reflection of waves from the nearer end of the string, the algebraic method becomes too lengthy and so it is not followed here.

It is found that, when the hammer strikes at a distance 100 cms. from one of the ends, the pressure terminates before the fourth reflection from the shorter end, and the third reflection from the remoter end overtakes the hammer. Again when the striking distance is 80 cms., the pressure terminates before the fifth reflection from the shorter end, and the third reflection from the remoter end overtakes the hammer. In the same way, for the striking lengths, 60 cms., 50 cms., 40 cms., 30 cms., and 25 cms., the pressure terminates before the fifth, sixth, seventh, eighth, and tenth reflections from the shorter end respectively, and the second reflection from the remoter end in each case overtakes the hammer. For the striking distance 20 and 15 cms., it is found that the pressure terminates before ninth and tenth reflections from the nearer end and any reflection from the remoter end overtake the hammer. It is found from figures (2) and (3) that the agreement between the theory and experiment is remarkable.

In the case of the elastic hammer we have got very complicated forms of the pressure function. So in very rare cases the duration of impact can be algebraically solved.

When the hammer strikes very near the end, so that $\frac{a}{c}$ is very small, the displacement of the struck-point is given by [*vide* eq. (23), Part II]

$$y_a = \frac{Ev_0c}{T}\left[A^2 e^{\alpha t} + \frac{A}{\nu} e^{\mu t} \sin(\nu t - \epsilon)\right]$$

where
$$A = \frac{1}{\sqrt{(\alpha-\mu)^2+\nu^2}}\,; \qquad \alpha = \frac{E}{2m}\cdot\frac{1}{\mu} + \frac{c}{a}\,;$$

$$\mu = -\frac{T}{2m_1c}\,; \qquad \nu = \sqrt{-\frac{2\mu c}{a} - \mu^2}\,;$$

$$m_1 = m_0 + \frac{Tm}{Ea}\,; \qquad m_0 = m + \frac{a\rho}{3}\,; \qquad \tan\epsilon = \frac{\nu}{\mu - \alpha},$$

and the corresponding pressure is given by [*vide* eq. (26.2), Part II],

$$\begin{aligned} -P = Eu &= \left\{\frac{T}{c}D + \frac{T}{a}\right\} y_a, \\ &= -m_1[2\mu D - (\mu^2+\nu^2)]y_a \\ &= m_1 v_0 c \frac{AE}{\nu T}\left[A\nu(\mu^2+\nu^2-2\mu\alpha)e^{\alpha t} - (\mu^2+\nu^2)e^{\mu t}\right. \\ &\qquad\left.\sin\left(\nu t - \epsilon + \tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2}\right)\right] \qquad \ldots (27) \end{aligned}$$

and in the usual way the approximate value of the duration of contact is obtained, as given by

$$\Phi = \frac{1}{\nu}\left\{\tan^{-1}\frac{\nu}{\mu-\alpha} - \tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2}\right\}. \qquad \ldots (81)$$

The expression representing the pressure functions in other cases of the elastic hammer have got three different forms depending upon the values of the elastic constant, *i.e.*, according as $\frac{E}{T} >=< \frac{16\rho}{m}$. Here graphical method may be helpfully adopted except when $\frac{E}{T} = \frac{16\rho}{m}$, where well-known Horner's method or Newton's method may also be followed for numerical solution. In every case, it is evident from the pressure equations that the pressure must not terminate during the first interval, except when $\frac{E}{T} < \frac{16\rho}{m}$. Here pressure exerted by the hammer is given by [*vide* eq. (40.1), Part II]

$$P_1 = 4\rho v_0 c \frac{\mu_1}{\nu_1} e^{-\mu_1 t} \sin \nu_1 t$$

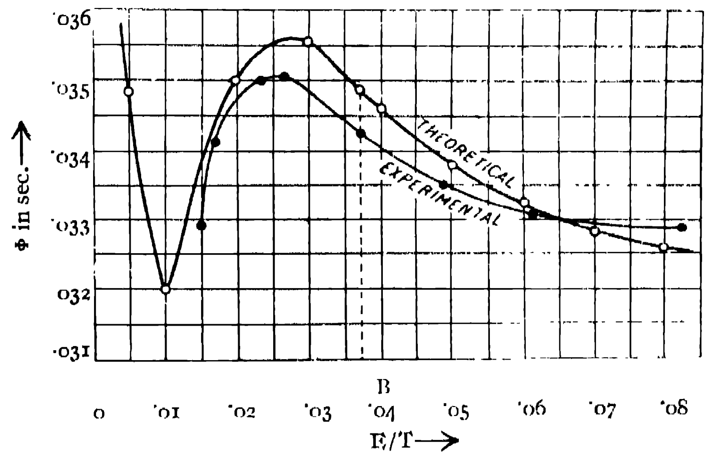
where

$$\mu_T = \frac{Ec}{2T}; \quad \nu_1 = \tfrac{1}{2}\sqrt{\frac{4E}{m} - \left(\frac{Ec}{2T}\right)^2}$$

and the duration of contact is given by

$$\Phi = \frac{2\pi}{\sqrt{\frac{4E}{m} - \left(\frac{Ec}{2T}\right)^2}}. \quad \ldots \quad (8a)$$

The *elastic constant E* of the hammer, which is included in the pressure function *is distinctly different from Young's Modulus* but it depends upon the same, and also upon the size and the shape of the hammer, and upon the length of contact. It is the force in dyne acting upon the length of contact for a short time to produce unit compression. The magnitude of such a force, when acting for a long time, measured in an ordinary statical method, is much less than the dynamical value, but these two sets of values may be interrelated.



The dynamical value of $\frac{E}{T}$ is ·03 when the corresponding statical is 100/T.

FIGURE 4

In order to test the theory of the elastic hammer the duration of contact of elastic hammer for different values of the elastic constant E and for a given mass 21·2 gms. striking at the mid-point of a string of length 600 cms. of linear density 0·05 gm./cm. stretched under tension 38·5 kilograms weight is calculated. The experimental values giving the variation of the duration of contact with the variation of the elastic constant of the hammer when strikes at mid-point of the string are taken from a paper of M. Ghosh.[4] The values of the elastic constants given in the paper are statical values, and are different from the dynamical values required by our formula. So the statical values are reduced to dynamical values by selection, and both are represented graphically in fig. 4. Here we find that pressure terminates during the second epoch. Φ corresponding to the point B in fig. 4, is calculated by the help of the pressure equation $P_2=0$ [*vide* eq. (47.2)], when $\frac{E}{T}=\frac{16\rho}{m}$ all the points on the right of B are calculated graphically by the help of $P_2=0$ [*vide* eq. (47.2)], when $\frac{E}{T}>\frac{16\rho}{m}$ and all the points on the left of B are calculated by the help of $P_2=0$ when $\frac{E}{T}<\frac{16\rho}{m}$. The agreement between the theory and the experiment is good.

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